Exact quantum S-matrix in the Liouville field theory

1995 J. Phys. A: Math. Gen. 28 5375

(http://iopscience.iop.org/0305-4470/28/18/025)

J. Phys. A: Math. Gen. 28 (1995) 5375–5384. Printed in the UK

Exact quantum S-matrix in the Liouville field theory

I M Khamitov†
Alkor Technologies Inc., PO Box 116, St Petersburg 199155, Russia

Received 23 February 1995

Abstract. Classical scattering in the Liouville field theory (LFT) is essentially finite-dimensional, and in some cases the classical S-matrix can be represented as a transformation of the Poisson group $SL(2,\mathbb{R})$. Motivated by this and using conjectural quantum analogues of some ingredients of the classical model, we find an exact quantum S-matrix without constructing the quantum LFT in full. The quantum S-matrix is explicitly represented as a transformation of the quantum group $SL_q(2,\mathbb{R})$, and for a particular implementation, the S-matrix is shown to be unitary (unitarily generated).

1. Introduction

It is widely believed that there is no scattering in the Liouville field theory (LFT) in an infinite volume [1]. This belief, however, finds no confirmation at the classical level [2, 3]. Moreover, for naturally chosen phase spaces of the model, any field configuration exhibits non-trivial scattering, and the classical S-matrix takes the relatively simple form of a transformation of the Poisson group $SL(2,\mathbb{R})$ [3]. So it is reasonable to expect that the quantum S-matrix is non-trivial and 'finite-dimensional', and can be found exactly in closed form without explicitly constructing the quantum LFT. This would be especially interesting, because LFT still resists a satisfactory quantization, although the explicit classical solution is known. In the present paper we speculate on a possible quantum generalization of the classical model and succeed in representing the quantum S-matrix in closed form as a transformation of the quantum group $SL_q(2,\mathbb{R})$.

In the remainder of this section we recall the basic constituents of the classical model, our notation being as close to those of [3, 4] as possible. In section 2 we propose a quantum generalization of the classical objects and carry out various checks. In particular, the asymptotic fields are introduced and their commutation relations are calculated. In section 3 the scattering transformation is derived, and it is shown that the S-matrix appears to be unitary (unitarily generated) in a particular implementation.

A general solution to the Liouville equation

$$\Phi_{tt} - \Phi_{xx} + 4\mathrm{e}^{2\Phi} = 0$$

which is an equation of motion for the LFT, has the form [4, 5]

$$\mathrm{e}^{-\Phi(t,x)} = \Omega_+(x^+)T\Omega_-(x^-). \tag{1.1}$$

† E-mail address: khamitov@alkor-a.spb.su

Here T is a 2×2 real matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $ad - bc = 1$, i.e. $T \in SL(2, \mathbb{R})$; $x^{\pm} = x \pm t$ are the cone coordinates on the two-dimensional Minkowski space; the 2-row $\Omega_+(x)$ and the 2-column $\Omega_-(x)$ are real and satisfy the Wronskian condition

$$\Omega_+(x) \otimes \Omega'_+(x)(1 - \mathcal{P}) = (0, 1, -1, 0) \qquad (1 - \mathcal{P})\Omega_-(x) \otimes \Omega'_-(x) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

where $\mathcal{P}$ is a permutation matrix, i.e.

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the case of a singular field Φ, the right-hand side of the equality (1.1) is not positive definite. The corresponding sign function is assumed to be incorporated in the exponential function of the left-hand side.

The phase space of the model, Poisson structure and Ω may be chosen so that the following Poisson brackets hold (see [4]):

$$\{T \overset{\otimes}{,} T\} = [r, T \otimes T] \qquad r = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

(the square brackets designate the commutator; this formula equips $SL(2, \mathbb{R})$ with the structure of the Poisson Lie group)

$$\begin{aligned} &\{\Omega_+(x) \overset{\otimes}{,} \Omega_+(y)\} = \Omega_+(x) \otimes \Omega_+(y) r(x - y) \\ &\{\Omega_-(x) \overset{\otimes}{,} \Omega_-(y)\} = -r(x - y)\Omega_-(x) \otimes \Omega_-(y) \\ &\{\Omega_+(x) \overset{\otimes}{,} \Omega_-(y)\} = 0 \qquad \{\Omega_\pm(x) \overset{\otimes}{,} T\} = 0 \end{aligned} \tag{1.3}$$

where

$$r(x) = \mathcal{P} r \mathcal{P} \theta(x) - r\theta(-x) \qquad \theta(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0. \end{cases}$$

All the brackets (1.2), (1.3) are invariant under the action of the Poisson group $SL(2, \mathbb{R})$. This property together with the piecewise constancy of $r(x)$ ensures the local commutativity of the fundamental field Φ [4].

2. Quantum analogues of the classical objects

The quantum analogue of (1.2) is well known [6, 7] and has the form

$$R_q T \otimes T = T \otimes T R_q \tag{2.1}$$

where

$$R_q = q^{-1/2}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad q = e^{i\hbar/2}.$$

Here $\hbar$ is the Planck constant and the matrix elements of T, as well as those of $\Omega_\pm$ below, are implied to be real (self-adjoint). We also assume that the quantum determinant of T is unity:

$$\det_q T \stackrel{\text{def}}{=} ad - qbc = da - q^{-1}bc = 1. \tag{2.2}$$

The quantum analogue of (1.3) looks as follows:

$$\begin{aligned}
&\Omega_+(x) \otimes \Omega_+(y) = \Omega_+(y) \otimes \Omega_+(x) R(x-y) \\
&\Omega_-(x) \otimes \Omega_-(y) = R^{-1}(x-y)\Omega_-(y) \otimes \Omega_-(x) \\
&\Omega_+(x) \otimes \Omega_-(y) = \Omega_-(y) \otimes \Omega_+(x) \\
&\Omega_+(x) \otimes T\mathcal{P} = T \otimes \Omega_+(x) \qquad \Omega_-(x) \otimes T = \mathcal{P}T \otimes \Omega_-(x)
\end{aligned} \tag{2.3}$$

where

$$R(x) = R_q^{-1}\theta(x) + R_q\theta(-x).$$

The commutation relations of the kind (2.3) were previously suggested in the context of the theory in a finite volume [8]. We assume that they are somehow implemented in the Hilbert space of the model in an infinite volume. The field $e^{-\Phi(t,x)}$ is given by the same formula as in the classical case, i.e. by (1.1). Note that we do not make any attempt here to define the fundamental field $\Phi(t, x)$ itself. A reasonable character of the assumptions made is supported by a number of checks that follow.

(1) In the classical limit $\hbar \to 0$, $(i/\hbar)[\cdot,\cdot] \to \{\cdot,\cdot\}$ the commutation relation (2.1) turns into (1.2) due to the expansion

$$\mathcal{P}R_q = 1 + i\hbar r + O(\hbar^2) \qquad \hbar \to 0.$$

Likewise, the commutation relations (2.3) turn into (1.3) due to the expansion

$$\mathcal{P}R(x) = 1 - i\hbar r(x) + O(\hbar^2) \qquad \hbar \to 0.$$

(2) The commutation relations (2.3) are self-consistent due to the Yang–Baxter equation

$$\begin{aligned}
1 \otimes R(x-y) \cdot R(x-z) \otimes 1 \cdot 1 \otimes R(y-z) \\
= R(y-z) \otimes 1 \cdot 1 \otimes R(x-z) \cdot R(x-y) \otimes 1
\end{aligned}$$

which looks like the Yang–Baxter equation with a spectral parameter, but, for different locations of x, y and z, it reduces to various forms of the same Yang–Baxter equation (for R_q) without a spectral parameter:

$$1 \otimes R_q \cdot R_q \otimes 1 \cdot 1 \otimes R_q = R_q \otimes 1 \cdot 1 \otimes R_q \cdot R_q \otimes 1.$$

This last equation expresses the self-consistency of (2.1).

(3) All the commutation relations (2.1), (2.3) are invariant under the action of the quantum group $SL_q(2, \mathbb{R})$. This property together with the piecewise constancy of $R(x)$ ensures the local commutativity of the field (1.1) (see the next point).

(4) The following calculation demonstrates the commutativity of the field (1.1) for the spacelike separation ($y^{\pm} = y \pm s$):

$$\begin{aligned}
e^{-\Phi(t,x)}e^{-\Phi(s,y)} &= \Omega_+(x^+)T\Omega_-(x^-) \cdot \Omega_+(y^+)T\Omega_-(y^-) \\
&= \Omega_+(x^+) \otimes \Omega_+(y^+) \cdot T \otimes T \cdot \Omega_-(x^-) \otimes \Omega_-(y^-) \\
&= \Omega_+(y^+) \otimes \Omega_+(x^+)R(x^+ - y^+)T \otimes TR^{-1}(x^- - y^-)\Omega_-(y^-) \otimes \Omega_-(x^-) \\
&= \Omega_+(y^+) \otimes \Omega_+(x^+) \cdot T \otimes T\, \underbrace{R(x^+ - y^+)R^{-1}(x^- - y^-)}_{=1}\, \Omega_-(y^-) \otimes \Omega_-(x^-) \\
&= \Omega_+(y^+)T\Omega_-(y^-) \otimes \Omega_+(x^+)T\Omega_-(x^-) = e^{-\Phi(s,y)}e^{-\Phi(t,x)}
\end{aligned}$$

(the underbraced quantity is equal to unity only for spacelike separation, i.e. for $(x^+ - y^+)(x^- - y^-) > 0$).

(5) The commutation relations (2.1), (2.3) are compatible with the reflections of space and time. This is ensured by the following properties of the R-matrix:

$$R_q^{\mathrm{T}} = R_q \qquad \sigma_1 \otimes \sigma_1 R_{q^{-1}} \sigma_1 \otimes \sigma_1 = R_q^{-1} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

See appendix A for details.

(6) A plausible form of the quantum Wronskian condition looks as follows:

$$\begin{aligned}
&{:}\Omega_+(x) \otimes \Omega'_+(x){:}P_q^{(-)} = \frac{q^{1/2}}{q + q^{-1}}(0, q^{-1}, -1, 0) \\
&P_q^{(-)}{:}\Omega_-(x) \otimes \Omega'_-(x){:} = \frac{q^{-1/2}}{q + q^{-1}} \begin{pmatrix} 0 \\ -1 \\ q \\ 0 \end{pmatrix}
\end{aligned} \tag{2.5}$$

where the colons designate some (unknown) regularization of the operator product with the coincident spatial points, and $P_q^{(-)}$ is a quantum antisymmetrization matrix, i.e.

$$P_q^{(-)} = \frac{q - q^{1/2}R_q}{q + q^{-1}} = \frac{1}{q + q^{-1}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

We believe that the condition (2.5) is not an essentially independent one but a consequence of the commutation relations (2.1), (2.3) in the case of irreducible representation. Nevertheless, this condition provides the normalization for the homogeneous equations (2.3). Apart from the usual reference to the classical limit, two arguments may be adduced in favour of (2.5). First, the equalities (2.5) are compatible with the reflections of space and time. Second, they are invariant under $SL_q(2,\mathbb{R})$, which may be verified by using the following relations:

$$P_q^{(-)}T\otimes T = P_q^{(-)}\det_q T \qquad T\otimes T P_q^{(-)} = P_q^{(-)}\det_q T. \tag{2.7}$$

Let us now introduce asymptotic fields by the same formulae as in the classical case [3]:

$$\mathrm{e}^{-A_{\mathrm{in}}(t,x)} = \Omega_+(x^+)T\begin{pmatrix}1\\0\end{pmatrix}a^{-1}(1,0)T\Omega_-(x^-) = \Omega_+(x^+)\widetilde{T}\Omega_-(x^-) \tag{2.8}$$

$$\mathrm{e}^{-A_{\mathrm{out}}(t,x)} = \Omega_+(x^+)T\begin{pmatrix}0\\1\end{pmatrix}d^{-1}(0,1)T\Omega_-(x^-) = \Omega_+(x^+)\stackrel{\approx}{T}\Omega_-(x^-) \tag{2.9}$$

where the diagonal matrix elements of T, a and d, are assumed invertible (see the discussion of this assumption in the next section), and the notation for two matrices is introduced:

$$\widetilde{T} = \begin{pmatrix}a & b\\ c & ca^{-1}b\end{pmatrix} \qquad \stackrel{\approx}{T} = \begin{pmatrix}bd^{-1}c & b\\ c & d\end{pmatrix}.$$

These matrices are real and satisfy the same commutation relation as T (2.1) with the same R_q, but with

$$\det_q \widetilde{T} = 0 \qquad \det_q \stackrel{\approx}{T} = 0 \tag{2.10}$$

(irrespective of the value of $\det_q T$). In particular, this means that the asymptotic fields are real and locally commutative. Moreover, we can calculate their commutation relations in full. To accomplish this we need one formula, which will be now derived. It follows from the definition of $P_q^{(-)}$ (2.6) and its property (2.7) that

$$R_qT\otimes T = T\otimes TR_q = q^{1/2}T\otimes T - q^{-1/2}(q+q^{-1})P_q^{(-)}\det_q T.$$

Thus, taking into account (2.10) we obtain

$$R_q\widetilde{T}\otimes\widetilde{T} = \widetilde{T}\otimes\widetilde{T}R_q = q^{1/2}\widetilde{T}\otimes\widetilde{T}$$

or equivalently

$$R(x)\widetilde{T}\otimes\widetilde{T} = \widetilde{T}\otimes\widetilde{T}R(x) = \left(q^{-1/2}\theta(x)+q^{1/2}\theta(-x)\right)\widetilde{T}\otimes\widetilde{T}. \tag{2.11}$$

The same is true for $\widetilde{T}$ replaced by $\stackrel{\approx}{T}$. The calculation of the commutation relation of the in-field (2.8) runs much the same as in the case of the field $\mathrm{e}^{-\Phi}$, see point (4) above, but now the underbraced quantity is multiplied by $\widetilde{T}\otimes\widetilde{T}$ from the left, and the relation (2.11) immediately leads to the following commutation relation:

$$\begin{aligned}\mathrm{e}^{-A_{\mathrm{in}}(t,x)}\mathrm{e}^{-A_{\mathrm{in}}(s,y)} &= \mathrm{e}^{-A_{\mathrm{in}}(s,y)}\mathrm{e}^{-A_{\mathrm{in}}(t,x)}\left(q^{-1/2}\theta(x^+-y^+)+q^{1/2}\theta(y^+-x^+)\right)\\ &\quad\times\left(q^{-1/2}\theta(x^--y^-)+q^{1/2}\theta(y^--x^-)\right)^{-1}\\ &= \mathrm{e}^{-A_{\mathrm{in}}(s,y)}\mathrm{e}^{-A_{\mathrm{in}}(t,x)}\begin{cases}q^{-1} & \text{if } (x-y)^2-(t-s)^2<0 \quad t>s\\ 1 & \text{if } (x-y)^2-(t-s)^2>0\\ q & \text{if } (x-y)^2-(t-s)^2<0 \quad t<s.\end{cases}\end{aligned} \tag{2.12}$$

In appendix B we give another derivation of (2.12), which is closer to the respective classical calculation [3] and may be useful for other models. Note that (2.12) is the very relation one could expect to obtain for $e^{-A_{in}}$ starting from the standard free-field commutation relation for A_{in}:

$$\frac{i}{\hbar}[A_{in}(t,x), A_{in}(s,y)] = \tfrac{1}{4}\text{sign}(x^+ - y^+) - \tfrac{1}{4}\text{sign}(x^- - y^-).$$

Using (2.11) with $\widetilde{T}$ replaced by $\widetilde{\widetilde{T}}$, we may prove the commutation relation (2.12) for the out-field as well.

The question as to whether or not the field $e^{-\Phi}$ (1.1) interpolates between the fields $e^{-A_{in}}$ and $e^{-A_{out}}$ defined by (2.8) and (2.9) is unanswerable without precise knowledge of the specific representation of the algebra (2.3). Here we only note that in the classical case the asymptotic condition for the exponents looks as follows:

$$e^{-\Phi(t,x+vt)} - e^{-A_{in}(t,x+vt)} = O(1) \qquad t \to -\infty \quad \text{for all } x, v \in \mathbb{R},\ |v| \leqslant 1.$$

Both exponents grow in this limit, so for the fields themselves, we have

$$\Phi(t, x+vt) - A_{in}(t, x+vt) \to 0 \qquad t \to -\infty \quad \text{for all } x, v \in \mathbb{R},\ |v| \leqslant 1.$$

3. Quantum S-matrix

By definition, the scattering transformation maps the in-field (2.8) onto the out-field (2.9). As may be readily noticed, such a mapping non-trivially transforms only T, leaving $\Omega_\pm$ unchanged (provided bc is invertible). Denote the restriction of the scattering transformation on T as S', then it looks as follows:

$$\begin{gathered} S'(T) = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \\ b' = b \qquad c' = c \qquad a' = \frac{qbc}{1+qbc}a \qquad d' = d\frac{1+qbc}{qbc}. \end{gathered} \tag{3.1}$$

The invertibility of $ad = 1 + qbc$ implied here and also assumed in section 2, is discussed below. It should be emphasized, however, that the non-invertibility of bc does not prevent the S-matrix from existing, but only means that it cannot be represented in such a simple form as (3.1). The formula (3.1) for the quantum LFT S-matrix is the main achievement of the paper.

Let us discuss the problem of invertibility of a, b, c, and d provided that they are implemented as self-adjoint operators in the Hilbert space $\mathcal{H}$. The commutation relation (2.1) in terms of the matrix elements of T looks as follows:

$$\begin{gathered} ab = qba \qquad ac = qca \qquad bd = qdb \qquad cd = qdc \\ bc = cb \qquad ad - da = (q - q^{-1})bc \end{gathered} \tag{3.2}$$

(recall that $|q| = 1$ and in addition we now assume $q \neq \pm 1$, i.e. $0 < |\hbar| < 2\pi$). First of all, a precise sense in which these commutation relations are understood must be specified. We shall assume that there exists a linear set D dense in $\mathcal{H}$ and such that:

(1) $D \subset D(a) \cap D(b) \cap D(c) \cap D(d)$;

(2) $aD \subset D$, $bD \subset D$, $cD \subset D$, and $dD \subset D$;

(3) for all $f \in D$, $abf = qbaf$, $acf = qcaf$, $bdf = qdbf$, $cdf = qdcf$, $bcf = cbf$ and $adf - daf = (q - q^{-1})bcf$;

(4) the operators a, b, c and d are essentially self-adjoint on D;

(5) $P_\lambda^{(z)} D \subset D$ $(z = a,\ b,\ c,\ d)$, where $P_\lambda^{(z)}$ is the orthogonal projector on the proper subspace of the operator z, corresponding to the eigenvalue λ.

Under these assumptions the Hilbert space $\mathcal{H}$ decomposes into an orthogonal sum:

$$\mathcal{H} = \bigoplus_{j=1}^{5} \mathcal{H}_j \tag{3.3}$$

where $\mathcal{H}_j$, $j = 1, 2, 3, 4, 5$ are invariant subspaces of the operators a, b, c, and d such that:

(1) in $\mathcal{H}_1$ the operators a, b, c, and d do not have a pointwise spectrum;

(2) in $\mathcal{H}_2$ the operators a, b and d do not have a pointwise spectrum, and $c = 0$;

(3) in $\mathcal{H}_3$ the operators a, c and d do not have a pointwise spectrum, and $b = 0$;

(4) in $\mathcal{H}_4$ the operators a and d are invertible, and $b = 0$, $c = 0$;

(5) in $\mathcal{H}_5$ $\det_q Tf = 0$ for all $f \in D$.

Obviously, the subspace $\mathcal{H}_5$ cannot appear in the Hilbert space of representation of the algebra (3.2) with $\det_q T = 1$. In particular, this means that in the representations with $\det_q T = 1$ the operators a and d are *always invertible*, thus, we may legitimately use expressions (2.8) and (2.9) for the asymptotic fields. It is interesting to compare this with the situation in the classical model, where a and d may vanish on a zero-measure portion of the phase space (in the case of singular field). For such exclusive field configurations the asymptotic fields possess non-d'Alembertian components [3].

In the case of an irreducible representation, the decomposition (3.3) consists of only a single summand. If it is $\mathcal{H}_4$, it must be one dimensional, so that a and $d = 1/a$ are nothing but real numbers. This is the only representation of (2.2), (3.2) by the finite-dimensional matrices, because in the spaces $\mathcal{H}_{1,2,3}$ the operators a and d possess a continuous spectrum. In the space $\mathcal{H}_1$ all the relations (3.2) are a consequence of

$$ab = qba. \tag{3.4}$$

Indeed, the combination $\varkappa = cb^{-1}$ commutes with everything, so in an irreducible representation it must be a non-zero real number. For given $\varkappa \in \mathbb{R}$, $\varkappa \neq 0$ and a and b obeying (3.4), introduce c and d as follows:

$$c = \varkappa b \qquad d = a^{-1}(1 + \varkappa q b^2). \tag{3.5}$$

Then (2.2) and (3.2) are satisfied. The relation (3.4) is the only independent one in the space $\mathcal{H}_2$ as well. This space may be regarded as the limiting case of $\mathcal{H}_1$ when $\varkappa = 0$. Likewise, the space $\mathcal{H}_3$ may be regarded as the limiting case of $\mathcal{H}_1$ when $\varkappa = \infty$, the only independent relation being $ac = qca$.

Only the space $\mathcal{H}_1$ meets the invertibility condition of bc needed for the scattering transformation to be represented in the form (3.1). Let us describe the S-matrix (3.1) in more detail for a particular representation of (3.2) of the type $\mathcal{H}_1$. Let $P = (\hbar/\mathrm{i})\mathrm{d}/\mathrm{d}x$ and $Q = x$ be the standard differentiation and multiplication operators in $\mathcal{H}_1 \equiv L^2(\mathbb{R})$. Define

$$a = \mathrm{e}^{-P} \qquad b = \mathrm{e}^{Q/2}.$$

These operators together with c and d given by (3.5) satisfy the relations (2.2) and (3.2), $\varkappa$ being a parameter of the representation. As a common domain D one can choose a set of functions of the kind $g(x)\exp(-\frac{1}{2}x^2)$, where $g(x)$ is an entire function of order not exceeding 1. *Provided* $\varkappa > 0$, the S-matrix (3.1) in this representation is *unitarily generated*, i.e.

$$a' = s^{-1}as \qquad b' = s^{-1}bs \qquad c' = s^{-1}cs \qquad d' = s^{-1}ds \tag{3.6}$$

where s is a multiplication operator: $sf(x) = s(x)f(x)$ for all $f \in L^2(\mathbb{R})$, the function $s(x)$ being given by

$$s(x) = \exp\left(\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty} \frac{\log \xi(\mathrm{e}^y)}{1+\exp\big(2\pi(x-y)/|\hbar|\big)}\,\mathrm{d}y\right) \qquad \xi(z) = \frac{\varkappa z}{1+\varkappa z}. \tag{3.7}$$

The unitary operator s is defined by (3.6) uniquely up to multiplying the function $s(x)$ by a unitary constant.

In the described representation of the algebra (3.2) we have $a > 0$, $b > 0$. The unitarity of the scattering requires that $\varkappa > 0$, and hence $c > 0$, $d > 0$ too. Recall that in the classical model the inequalities $a > 0$, $d > 0$, $b \geqslant 0$, $c \geqslant 0$ (or equivalently $a < 0$, $d < 0$, $b \leqslant 0$, $c \leqslant 0$) constitute a necessary condition for the local fields to be non-singular [4]. At the same time, there are difficulties in Hamiltonian interpretation of the singular asymptotic fields [3]. So the necessity to choose $\varkappa > 0$ may indicate that the quantum analogues of the singular fields are ill-defined, at least some of them.

A similar representation of the algebra (3.2) in $L^2(\mathbb{R})$ with $\varkappa = 1$ was described in [6]. In that representation the matrix T looks a bit more symmetric:

$$T = \begin{pmatrix} \mathrm{e}^{-P/2}\sqrt{1+\varkappa\mathrm{e}^{Q}}\,\mathrm{e}^{-P/2} & \mathrm{e}^{Q/2} \\ \varkappa\mathrm{e}^{Q/2} & \mathrm{e}^{P/2}\sqrt{1+\varkappa\mathrm{e}^{Q}}\,\mathrm{e}^{P/2} \end{pmatrix}.$$

This implementation of the algebra (3.2) is unitarily equivalent to that described above, and may be obtained from the latter by the similarity transformation $(\cdot) \to u^{-1}(\cdot)u$, the operator u being multiplication by the function

$$u(x) = \exp\left(-\frac{\mathrm{i}}{2\hbar}\int_{-\infty}^{+\infty} \frac{\log(1+\varkappa\mathrm{e}^y)}{\exp\big(2\pi(x-y)/|\hbar|\big)+1}\,\mathrm{d}y\right).$$

The S-matrix in this representation is described by the same formulae (3.6), (3.7).

Appendix A

The time reflection τ is an antilinear automorphism of the algebra (2.1)–(2.3) that acts as follows:

$$\Phi^{\tau}(t,x) = \Phi(-t,x) \qquad T^{\tau} = \sigma_1 T^{\mathrm{T}}\sigma_1$$
$$\Omega_+^{\tau}(x) = \Omega_-^{\mathrm{T}}(x)\sigma_1 \qquad \Omega_-^{\tau}(x) = \sigma_1\Omega_+^{\mathrm{T}}(x).$$

That τ is an antilinear automorphism is ensured by the second equality in (2.4).

The space reflection σ is a linear automorphism of the algebra (2.1)–(2.3) that acts as follows:

$$\Phi^{\sigma}(t,x)=\Phi(t,-x) \qquad T^{\sigma}=(-1)^{n_-}T^{\mathrm{T}}(-1)^{n_+}$$
$$\Omega^{\sigma}_{+}(x)=\Omega^{\mathrm{T}}_{-}(-x)(-1)^{n_-} \qquad \Omega^{\sigma}_{-}(x)=(-1)^{n_+}\Omega^{\mathrm{T}}_{+}(-x)$$

where, on the classical level, n_+ is the number of eigenvalues of the zero-energy Schrödinger equation whose pair of independent solutions is $\Omega_+(x)$. n_- has an analogous meaning. In the quantum model, $n_\pm$ are supposedly self-adjoint operators with a discrete spectrum, which commute with everything, so in an irreducible representation they must be non-negative integers. The condition $n_\pm = 0$ is necessary for the local fields to be non-singular. That σ is a linear automorphism is ensured by the first equality in (2.4).

Appendix B

Here we give another derivation of the commutation relation of the in-field. The column $\binom{1}{0}\otimes\binom{1}{0}$ and the row $(1,0)\otimes(1,0)$ are eigenvectors of the matrices $R(x)$ and $R^{-1}(x)$, i.e.

$$\begin{aligned} R(x)\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}&=\left(q^{-1/2}\theta(x)+q^{1/2}\theta(-x)\right)\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\\ (1,0)\otimes(1,0)R^{-1}(x)&=\left(q^{-1/2}\theta(x)+q^{1/2}\theta(-x)\right)^{-1}(1,0)\otimes(1,0). \end{aligned} \tag{B.1}$$

The following chain of transformations is nothing but a straightforward quantum generalization of the respective classical calculation:

$$\mathrm{e}^{-A_{\mathrm{in}}(t,x)}\mathrm{e}^{-A_{\mathrm{in}}(s,y)}=\Omega_+(x^+)T\begin{pmatrix}1\\0\end{pmatrix}\underbrace{a^{-1}(1,0)T\Omega_-(x^-)}\cdot\underbrace{\Omega_+(y^+)T\begin{pmatrix}1\\0\end{pmatrix}a^{-1}}(1,0)T\Omega_-(y^-)$$

(the underbraced expressions, $(1,a^{-1}b)\Omega_-(x^-)$ and $\Omega_+(y^+)\binom{1}{ca^{-1}}$, are commutative because $[a^{-1}b, ca^{-1}]=0$, as follows from (2.1), so we may interchange them)

$$\begin{aligned} &=\left(\Omega_+(x^+)T\begin{pmatrix}1\\0\end{pmatrix}\cdot\Omega_+(y^+)T\begin{pmatrix}1\\0\end{pmatrix}\right)a^{-1}\\ &\qquad\times a^{-1}\Big((1,0)T\Omega_-(x^-)\cdot(1,0)T\Omega_-(y^-)\Big)\\ &=\left(\Omega_+(x^+)T\otimes\Omega_+(y^+)T\cdot\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right)a^{-1}\\ &\qquad\times a^{-1}\Big((1,0)\otimes(1,0)\cdot T\Omega_-(x^-)\otimes T\Omega_-(y^-)\Big)\\ &=\left(\Omega_+(y^+)T\otimes\Omega_+(x^+)T\,R(x^+-y^+)\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right)a^{-1}\\ &\qquad\times a^{-1}\Big((1,0)\otimes(1,0)R^{-1}(x^--y^-)T\Omega_-(y^-)\otimes T\Omega_-(x^-)\Big) \end{aligned}$$

(make use of (B.1))

$$= \Omega_+(y^+)T\begin{pmatrix}1\\0\end{pmatrix}\cdot\underbrace{\Omega_+(x^+)T\begin{pmatrix}1\\0\end{pmatrix}a^{-1}}\cdot\underbrace{a^{-1}(1,0)T\Omega_-(y^-)}\cdot(1,0)T\Omega_-(x^-)$$
$$\times\left(q^{-1/2}\theta(x^+-y^+)+q^{1/2}\theta(y^+-x^+)\right)$$
$$\times\left(q^{-1/2}\theta(x^--y^-)+q^{1/2}\theta(y^--x^-)\right)^{-1}$$

(interchange the underbraced quantities)

$$= \mathrm{e}^{-A_{\mathrm{in}}(s,y)}\mathrm{e}^{-A_{\mathrm{in}}(t,x)}\left(q^{-1/2}\theta(x^+-y^+)+q^{1/2}\theta(y^+-x^+)\right)$$
$$\times\left(q^{-1/2}\theta(x^--y^-)+q^{1/2}\theta(y^--x^-)\right)^{-1}.$$

Putting the ends of this chain together, we obtain (2.12).

References

[1] Thorn C B 1983 *Phys. Lett.* **128B** 207
D'Hoker E, Freedman D Z and Jackiw R 1983 *Phys. Rev.* D **28** 2583
Yoneya T 1984 *Phys. Lett.* **148B** 111

[2] Dzhordzhadze G P 1979 *Teor. Mat. Fiz.* **41** 33 (English transl. 1979 *Theor. Math. Phys.* **41** 867)

[3] Khamitov I M 1994 *J. Phys. A: Math. Gen.* **27** 7217

[4] Khamitov I M 1994 *J. Phys. A: Math. Gen.* **27** 955

[5] Pogrebkov A K 1979 *Dokl. Akad. Nauk SSSR* **244** 873 (English transl. 1979 *Sov. Phys. Dokl.* **24** 105)

[6] Faddeev L D and Takhtajan L A 1986 *Field Theory, Quantum Gravity and Strings* (Lecture Notes in Physics **246**) (Berlin: Springer) p 166

[7] Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1989 *Algebra i Analiz* **1** 178 (in Russian)

[8] Gervais J-L and Neveu A 1984 *Nucl. Phys.* B **238** 125
Babelon O 1988 *Phys. Lett.* **215B** 523
Faddeev L D 1990 *Commun. Math. Phys.* **132** 131